MATH 2050C Lecture on 3/11/2020

Last time ...... subsequences (§3.4)

sequence $(x_n) = (x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8 \cdots x_n \cdots)$ labelled by $n \in \mathbb{N}$

sub-sequence $(x_{n_k}) = (x_1\ x_3\ x_5\ x_7 \cdots \cdots x_{n_k} \cdots)$ labelled by $k \in \mathbb{N}$

E.g.) $n_1 = 1,\ n_2 = 3,\ n_3 = 5,\ n_4 = 7, \ldots, n_k = 2k-1$

Q: How to determine the convergence / divergence of (x_n) ?

Summary so far

- **Monotone Convergence Thm (MCT)**

  Assume (x_n) is monotone. Then (x_n) convergent $\overset{\text{Thm 1}\ (\Rightarrow)}{\Longleftrightarrow}$ (x_n) bdd.

  Remark: This is useful to show convergence.

- **Thm 1**: (x_n) convergent $\Rightarrow$ (x_n) bdd
- **Thm 2**: (x_n) convergent $\Rightarrow$ $\lim\limits_{k\to\infty}(x_{n_k}) = \lim\limits_{n\to\infty}(x_n)$ for all subseq. (x_{n_k}) of (x_n)

  Remark: These cannot be used to prove convergence. BuT useful to prove (x_n) is divergent.

  Thm 1': (x_n) unbdd $\Rightarrow$ (x_n) divergent

  Thm 2'a: $\exists$ subseq's (x_{n_k}) and $(x_{n_{k'}})$ of (x_n) s.t. $\Rightarrow$ (x_n) divergent

  $$\lim_{k\to\infty}(x_{n_k}) \neq \lim_{k'\to\infty}(x_{n_{k'}})$$

  exist BUT different

  Thm 2'b: $\exists$ sub seq. (x_{n_k}) of (x_n) which is divergent $\Rightarrow$ (x_n) divergent

Examples: (i) $(x_n) = (n)$ unbdd $\Rightarrow$ divergent by Thm 1'

(ii) $(x_n) = ((-1)^n)$ has subseq's

$$(x_{2k-1}) = (-1, -1, -1, \ldots) \to -1$$
$$(x_{2k'}) = (1, 1, 1, 1, \ldots) \to 1$$

$-1 \neq 1$ $\Rightarrow$ (x_n) divergent by Thm 2'a

(iii) $(x_n) = \left(\sin\left(\frac{n\pi}{2}\right)\right)$ has subseq.

$(x_{2k-1}) = (1, -1, 1, -1, \dots)$ divergent $\xRightarrow{\text{by Thm 2'b}}$ (x_n) divergent

Recall:

- (x_n) convergent $\Leftrightarrow$ (x_n) converges to **some** $x \in \mathbb{R}$
- (x_n) divergent $\Leftrightarrow$ (x_n) **DOES NOT** converge to **ANY** $x \in \mathbb{R}$

Prop: Fix $x \in \mathbb{R}$. Then (x_n) does not converge to x

$\Leftrightarrow$ $\exists\, \varepsilon_0 > 0$ and some subseq. (x_{n_k}) of (x_n) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \qquad \forall k \in \mathbb{N}$$

E.g.

$(x_n) = (1, 1, 1, 2, 1, 3, \dots$
$\dots 1, n, 1, n+1, \dots)$

Remark: This Prop. is useful to show (x_n) does not converge to a given x.
But sometimes difficult to prove divergence ($\because$ check all possible x).

Proof: (basically MATH 1050)

$\sim\big((x_n)$ converges to x $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists K = K(\varepsilon) \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ $\forall n \geq K\big)$

(x_n) does not converge to x $\Leftrightarrow$ $\exists\, \varepsilon_0 > 0$, $\forall K \in \mathbb{N}$, $\exists\, n_k \geq K$ s.t.
$|x_{n_k} - x| \geq \varepsilon_0$

"$\Rightarrow$" For this $\varepsilon_0 > 0$, we define the subseq. (x_{n_k}) as follows:

Take $K = 1$, then $\exists\, n_1 \geq 1$ s.t. $|x_{n_1} - x| \geq \varepsilon_0$

Take $K = n_1 + 1$, then $\exists\, n_2 \geq n_1 + 1$ s.t. $|x_{n_2} - x| \geq \varepsilon_0$

Take $K = n_2 + 1$, then $\exists\, n_3 \geq n_2 + 1$ s.t. $|x_{n_3} - x| \geq \varepsilon_0$

Inductively, we get $n_1 < n_2 < n_3 < \dots < n_k < \dots$ s.t.
the subseq. (x_{n_k}) satisfies $|x_{n_k} - x| \geq \varepsilon_0$ $\forall k \in \mathbb{N}$.

"$\Leftarrow$" Exercise.

$\blacksquare$

Remember: MCT: (x_n) bdd AND monotone $\Rightarrow$ (x_n) convergent

Q: What about only bdd?

A: Bolzano-Weierstrauss Thm (BWT): (x_n) bdd $\Rightarrow$ $\exists$ subseq. (x_{n_k}) of (x_n) which is convergent.

E.g.) $(x_n) = ((-1)^n)$ bdd, NOT convergent

Yet, $(x_{2k-1}) = (-1, -1, -1, \dots) \to -1$

$(x_{2k}) = (1, 1, 1, \dots) \longrightarrow 1$